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CORRECTION OF TRANSIENT THERMOCOUPLE TRANSIENT MEASUREMENTS IN HEAT-CONDUCTING SOLIDS

PART II

THE CALCULATION OF TRANSIENT HEAT FLUXES USING THE INVERSE CONVOLUTION

Prepared by

James V. Beck

RESEARCH AND ADVANCED DEVELOPMENT DIVISION
AVCO CORPORATION
Wilmington, Massachusetts

Technical Report

RAD-TR-7-60-38 (Part II)

Contract AF04 (647) -258

THIS REPORT WAS PREPARED IN ACCORDANCE WITH AIR FORCE CONTRACT NUMBER AF04(647)-258. IT IS SUBMITTED IN PARTIAL FULFILLMENT OF THE CONTRACT AND IN ACCORDANCE WITH AFBM EXHIBIT 58-1 (PARAGRAPH 4.2.2).

30 March 1961

Prepared for

AIR FORCE BALLISTIC MISSILE DIVISION
AIR RESEARCH AND DEVELOPMENT COMMAND
UNITED STATES AIR FORCE
Inglewood, California

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Project WS133A

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30 March 1961

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ABSTRACT

A new calculation method is presented to obtain the surface heat flux from <u>one</u> interior temperature measurement in a heat conducting solid possessing constant thermal properties.

The new method of calculating the flux involves the numerical inversion of the convolution integral which is a Volterra equation of the first kind. The method employs a least squares procedure.

A comparison with the older numerical inversion procedure of Stolz is given. The new procedure is shown to be stable for smaller steps than Stolz's method and also to be markedly less oscillatory. The new procedure is recommended over that used by Stolz.

SUMMARY

Need has existed for some time for an improved method to calculate heat fluxes at the surface of a solid from one transient interior temperature measurement. One need for this heat flux is for the correction of thermocouple temperature readings as discussed in Part I of this report.[1] This present report, Part II, has been written to supply such a method.

The determination of surface flux from an interior temperature however is basically difficult due to the physics of the problem. The surface flux can be calculated only with an associated degree of uncertainty using any numerical method based on measured data. This uncertainty results from the damping in the temperature response due to the heat capacity of the solid. The damping causes a loss of information of the thermocouple since it measures temperatures to only a finite degree of accuracy. Klamkin has also examined the difficulties associated with this problem.[3,9]

The heat flux was calculated by numerically inverting the convolution integral. The convolution may be given in the form

$$T(x_1, t) = \int_0^t \dot{q}(0,\lambda) \frac{\partial \phi(x_1, t-\lambda)}{\partial t} d\lambda , \qquad \text{(A)}$$

where $T(x_1, t)$ is the measured temperature at point x_1; $\phi(x_1, t)$ is the temperature at point x_1 due to a unit step rise of the surface heat flux and $\dot{q}(0,t)$ is the transient surface heat flux. The convolution applies only for linear conduction problems (i. e. , the thermal properties must be constant with temperature; there is no internal radiation and so on). This was, however, the only case considered in Part I for several reasons.

When T and ϕ are known in equation (A) and $\dot{q}$ is to be determined, equation (A) is mathematically classed as a Volterra equation of the first kind. For convenience equation (A) is referred to in the report as the "inverse convolution." A direct numerical inversion of equation (A) has been given by Stolz,[2] and a similar method has been used by the author.[1] This recurrence formulation, see equation (3), however possesses certain undesirable characteristics; it gives $\dot{q}$ values which oscillate in time for "large" time steps and is unstable for "small" time steps. Because of these undesirable features an improved method of numerically calculating the inverse convolution was needed.

The new inverse convolution method is also a recurrence formulation and incorporates the following features: 1) One or more measured temperatures at k future times ($t + \Delta t, t + 2\Delta t, \ldots t + k\Delta t$) may be used while calculating the heat

flux at time t ; 2) The form of the heat flux from time t to time $t + k\Delta t$ is assumed to be a constant (if k was chosen to be equal to or greater than unity), to be linear with time (if $k \geq 2$) and so on; 3) The most probable $\dot{q}$ at time t is then calculated using the method of least squares. A large number of possible recurrence relations appear in this family depending on the choice of k and the form of $\dot{q}$. The one which has been examined in greatest detail is the simplest: $k = 1$ and $\dot{q}$ = constant.

This recurrence formulation is given by

$$\dot{q}_M = \frac{1}{C_2}\left[{}_eT_M + C_1 \, {}_eT_{M+1} - \sum_{n=1}^{M-1} \dot{q}_n \bar{\phi}_{M-n} \right] , \qquad \text{(B)}$$

where

$$C_1 = 1 + \frac{\Delta\phi_1}{\Delta\phi_0} ,$$

$$C_2 = \Delta\phi_0 (1 + C_1^2) ,$$

$$\bar{\phi}_{M-n} = \Delta\phi_{M-n} + C_1 \Delta\phi_{M-n+1} .$$

More complex equations are given by equations (20g) and (22F). Equation (B) has been programmed by the Mathematics Section and is called Problem 985.02. A comparison between the Stolz method and equation (B) is given for several examples. The comparison indicates: 1) that with equation (B) calculations may be made with time steps as small as 20 percent of the smallest allowable using the Stolz method, and 2) that the amplitude of the oscillations in the calculations for $\dot{q}$ are at least an order of magnitude smaller using the new method than with Stolz's. As a result of these observations, it is recommended that the method of calculating surface heat flux described be used in preference over the method of Stolz.

The inverse convolution may be used in connection with a number of other heat conduction problems in addition to calculating surface heat flux. A number of these uses are suggested. One use is the determination of certain kernel functions which may be used in correction of thermocouple temperature measurements as described in Part I[1] and Part III.[11]

CONTENTS

ILLUSTRATIONS

NOMENCLATURE

Symbol	Description	Dimension
$a_i = \sum_{j=0}^{i} \Delta\phi_j$	Dimensionless constant	
$b_\ell = \sum_{i=0}^{\ell} a_i$	Dimensionless constant	
c_p	Specific heat	Btu/lb-°F
H_{12}	Kernel function, equation (4b)	
H_{21}	Kernel function, equation (4a)	
k	Thermal conductivity	Btu/ft-hr-°F
$M = t/\Delta t$	Dimensionless time index	
$\dot{q}$	Heat flux	Btu/ft^2-hr
r	Radial distance	Ft
R	Characteristic length	Ft
t	Time	Hr
T	Temperature	°F
$_eT$	Measured temperature	°F
x	Axial distance	Ft
$\alpha = k/\rho c_p$	Thermal diffusivity	ft^2/hr
θ_1	Temperature at point 1	°F
θ_2	Temperature at point 2	°F

NOMENCLATURE (Cont'd)

Symbol	Description	Dimension
λ	Dummy time variable	Hr
ρ	Density	lb/ft^3
$\tau = \frac{\alpha t}{R^2}$	Dimensionless time	
ϕ	Temperature response due to unit step rise in heat flux	°F

I. INTRODUCTION

In evaluating new heat-shield materials, determining re-entry flight test conditions, testing of rocket nozzles, and numerous other situations in which the surface heat flux and temperatures are high, it is necessary to calculate the transient surface heating rates. The calculation of a surface heat-flux given one transient temperature measurement in the interior of a heat conducting solid, however, has proven to be quite difficult. This difficulty is related to damping of the system which is caused by the heat capacity of the solid. One approach to calculate the surface heat flux has been described in Part I of this series.[1] It is an approach which inverts an integral equation--the convolution. The method of inverting this equation used in Part I is similar to that used by Stolz.[2] This approach applies only for the case in which the thermal properties are constant with temperature.

Another method of calculating the surface heat flux which has been used at Avco employs a finite difference formulation of the governing partial differential equation

$$\frac{\partial}{\partial x} k \frac{\partial T}{\partial x} = \rho c_p \frac{\partial T}{\partial t} .$$

This has been discussed by Klamkin[3] and programmed by Morriello and Brodeur.[4] This finite-difference method may be used for temperature-variable thermal properties.

The results of the Stolz and Klamkin methods are similar since in each case the results tend to oscillate even for "large" time steps and are unstable for "small" time steps. Both are also similar in that no future information is used to calculate the flux at the present time (as is used in the method to be described in this paper). Oscillations and instabilities tend to appear at inconveniently large time steps using either of these two methods. For this reason, a method with less oscillatory characteristics and which was also more stable was needed. This report describes a method which is based on the "inverse" convolution as does Stolz's but possesses more desirable characteristics. Though this method also applies only for constant thermal properties, it may be extended to treat temperature-variable thermal properties by an analogous finite-difference formulation.

The purposes of this paper are: 1) to present an improved inverse convolution procedure; 2) to compare the proposed procedure with Stolz's; and 3) to suggest further uses of the inverse convolution.

II. CONCLUSIONS

1. An improved method of calculating the heat flux at a surface of a heat conducting solid from one interior temperature measurement is given. This method involves the numerical inversion of the convolution integral and thus is applicable solely for linear problems. The method is the recurrence formulation given by equation 13.

2. This new method is shown to be much superior to that proposed by Stolz.[2] It permits time increments which may be as small as 20 percent of the minimum values associated with the Stolz procedure and it also yields markedly less oscillatory results than Stolz's method.

3. There are a number of uses of the inverse convolution in addition to calculation of surface fluxes. These include calculations of certain kernel functions[1] which may be used in the correction of thermocouple measurements.

4. The basic ideas used in numerically inverting the convolution may be applied to the numerical solution of non-linear conduction problems (i. e., temperature-dependent thermal properties).

III. RECOMMENDATION

The method of calculation of surface heat flux from one interior temperature described in this paper should be used in preference to Stolz's method. Since this method is only directly applicable to treat materials with constant thermal properties, it is recommended that work should be continued on the extension of this method to treat cases with temperature-variable thermal properties.

IV. DISCUSSION

One method of calculating the surface heat flux from an interior temperature measurement uses the convolution in the form

$$T(x_1, t) = \int_0^t \dot{q}(0, \lambda) \frac{\partial \phi(x_1, t-\lambda)}{\partial t} \, d\lambda \quad , \qquad (1)$$

where $T(x_1, t)$ is the transient temperature measurement at the interior point (x_1) in a one-dimensional solid due to the time-variable surface heat flux $\dot{q}(0, t)$. $\phi(x_1, t)$ is temperature response at point (x_1) for a unit step rise of the surface heat flux. In many problems the temperature T is calculated for known values of $\dot{q}$ and ϕ; this is a problem involving the forward convolution. The forward convolution involves an integration which may be performed exactly or numerically. A numerical formulation is given and discussed in Part I. [1]

When equation (1) is used to calculate the heat flux $\dot{q}$, the temperature T and ϕ are known. In this case, the unknown $\dot{q}$ appears under the integral sign; integral equations of this type are called Volterra equations of the first kind. If equation (1) is used to calculate $\dot{q}$, then equation (1) is used in an inverse fashion and thus is called the "inverse" convolution.

A number of methods of numerically solving integral equations have been known for some time. [5] These methods, however, do not appear to be directly applicable to the inverse convolution. A method of obtaining the inverse convolution using the Laplace transform and inverse Laplace transform has been suggested. [6] It may also be obtained by reducing the convolution to a Volterra equation of the second kind which may be solved by a method of successive substitutions. [7] A more direct procedure, however, would be preferred; such a method of numerically calculating $\dot{q}$ in equation (1) has recently been given by Stolz. [2] A similar method of solution has been developed by the author and has been programmed. [1] (It is called Problem 793.02 by the Mathematics Section.) It possesses, however, certain characteristics which are undesirable.

A. STOLZ INVERSE CONVOLUTION METHOD

This inverse convolution program, Problem 793.02, will now be described before giving an improved procedure. Equation (1) may be evaluated numerically by using the trapezoidal rule to evaluate the integral. The result is

$$T_M = \sum_{n=1}^{M-1} \dot{q}_n \Delta\phi_{M-n} + \dot{q}_M \Delta\phi_0 \quad , \tag{2a}$$

for time $t = M\Delta t$. The notation used in equation (2a) is

$$T_M = T(x_1,t) = T(x_1, M\Delta t) \quad , \tag{2b}$$

$$\frac{\Delta\phi_{M-n}}{\Delta t} = \frac{\phi(x_1,(M-n+1)\Delta t) - \phi(x_1,(M-n)\Delta t)}{\Delta t} \approx \frac{\partial\phi(x_1,(M-n)\Delta t)}{\partial t} \quad , \tag{2c}$$

$$\dot{q}_n = \dot{q}\left(x_0, \left(n - \frac{1}{2}\right)\Delta t\right) \quad . \tag{2d}$$

In the integration Δt is made equal to $\Delta\lambda$.

The heat flux at time $(M - 1/2)\Delta t$ is obtained by solving equation (2a) for $\dot{q}_M$. The temperature T at time $M\Delta t$ is used but no other future temperature is used. Solving equation (2a) for $\dot{q}_M$ gives the recurrence formula

$$\dot{q}_M = \frac{1}{\Delta\phi_0}\left[T_M - \sum_{n=1}^{M-1} \dot{q}_n \Delta\phi_{M-n}\right] \quad . \tag{3}$$

A stability analysis of equation (3) has been given by Elrod.[8] This equation is used to calculate $\dot{q}$ in Problem 793.02.

The input required for this program includes a table of the temperature T versus time t and a table of ϕ versus t. For values of T and ϕ which are required between tabulated values of t, the program uses linear interpolation.

An example of the use of equation (3) to calculate $\dot{q}$ will be given. These results will be later used to compare with an improved inverse convolution procedure. Consider the semi-infinite body which is heated by the constant heat input $\dot{q}_0$ and is shown by figure 1. The equations to give the exact temperature at points 1 and 2 are given in Appendix A along with an equation to give the heat flux at 1. The problem is now to calculate a kernel function relating these two points (see

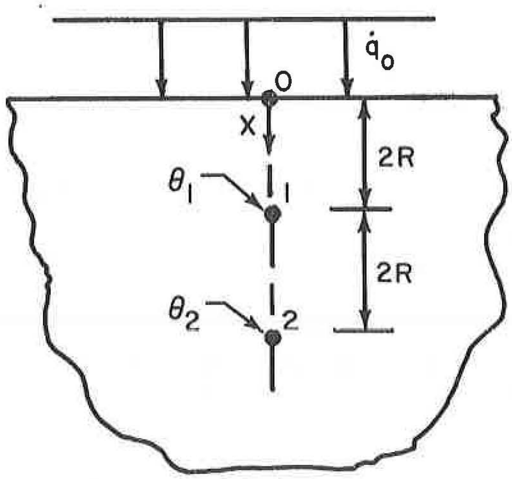

Figure 1 GEOMETRY FOR EXAMPLE OF SEMI-INFINITE BODY HEATED WITH A CONSTANT HEAT FLUX
61-1650

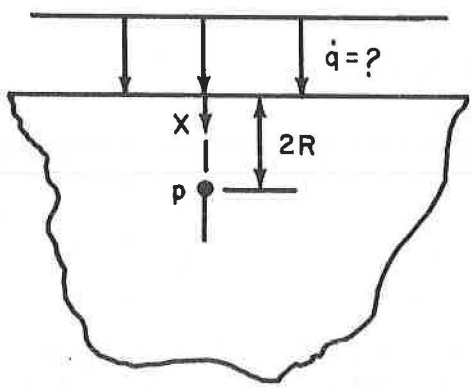

Figure 2 GEOMETRY FOR EXAMPLE OF SEMI-INFINITE BODY HEATED WITH A TIME-VARIABLE HEAT FLUX
61-1650

Part I). Two kernel functions will be found corresponding to

$$\theta_2(t) = \int_0^t H_{21}(\lambda) \frac{\partial \theta_1(t-\lambda)}{\partial t} d\lambda , \tag{4a}$$

and

$$\theta_1(t) = \int_0^t H_{12}(\lambda) \frac{\partial \theta_2(t-\lambda)}{\partial t} d\lambda . \tag{4b}$$

The exact result for H_{21} is given by[9]

$$H_{21} = \text{erfc} \frac{1}{\sqrt{\tau}} , \tag{4c}$$

where $\tau = at/R^2$ and points 1 and 2 are 2R apart. H_{12} on the other hand has been shown by Klamkin[9] not to exist in a strict mathematical sense. Values for H_{12} may, however, be obtained numerically from equation (4b). These values can not be the "true" values because these do not exist. At the same time it should be mentioned that although H_{12} does not exist in a mathematical sense yet the calculated values of H_{12} may be utilized for times greater than a predictable minimum value in certain cases. (See reference 9.)

The great utility of H_{21} (and H_{12} in certain cases) is that the temperature at the points 1 and 2 are related for <u>any</u> time-variable surface flux for a given geometry. The forward convolution with its "nice" properties might then be used rather than the inverse convolution.

The convolution equation given by equation (4a) has another physical meaning. Consider the example shown by figure 2. The temperature response at point "p" is given by $\theta_2(t)$ and it is required to find $\dot{q}$. The inverse convolution may be used to calculate $\dot{q}$ if the temperature response is known at 2R below the heated surface for a unit step heat flux. This temperature response in this case happens to be $\theta_1(t)$. The value of $\dot{q}$ required is given by equation (4c). The temperature at $z = 0$ (figure 2) may then be found by using the $\dot{q}$ and the temperature at $z = 0$ for a unit constant heat flux; it is found to be θ_1.

One might now try to give a physical picture of equation (4b); there is none, however.

The kernel functions H_{21} and H_{12} have been evaluated numerically and are shown by figures 3 and 4. The value of θ_1 and θ_2 were given to four significant figures and were given at all required times so that linear interpolation between given values was not employed.

The $\Delta\tau = 1$ calculation for H_{21} (figure 3) possesses oscillations which grow in time and become unbounded. The $\Delta\tau = 2$ calculation also has oscillations; these oscillations decrease with time, however. For all large time steps than $\Delta\tau = 2$ the results are stable and for all calculations with $\Delta\tau$ less than 1, the results are unstable. Table I gives the numerical values for a few time steps for $\Delta\tau = 1$ and 2. A comparison is made for each case with the exact values of H_{21}. In both cases the percent error alternates in sign from time step to time step. For the $\Delta\tau = 1$ case, the percent error first decreases and then begins to increase. For the $\Delta\tau = 2$ case, the error still alternates in sign but decreases in magnitude from step to step.

TABLE I

RESULTS OF CALCULATION FOR H_{21} USING PROBLEM 793.02 DIGITAL PROGRAM (STOLZ METHOD)

$\Delta\tau = 1.0$				
τ	H_{21} (exact)	H_{21}	$H_{21} - H_{21}$(ex.)	$\frac{H_{21} - H_{21}\text{(ex.)}}{H_{21}\text{(ex.)}}$ %
0.5	0.045	0.019	-0.026	-58
1.5	0.248	0.294	+0.046	+19
2.5	0.371	0.321	-0.050	-13
3.5	0.449	0.515	+0.016	+ 3.6
4.5	0.505	0.427	-0.078	-15
5.5	0.550	0.644	+0.094	+17
6.5	0.578	0.461	-0.117	-20
$\Delta\tau = 2.0$				
τ	H_{21} (exact)	H_{21}	$H_{21} - H_{21}$(ex.)	$\frac{H_{21} - H_{21}\text{(ex.)}}{H_{21}\text{(ex.)}}$ %
1.0	0.1573	0.1020	-0.0553	-35
3.0	0.4142	0.4606	+0.0464	+11
5.0	0.5271	0.5093	-0.0178	- 3.4
7.0	0.5930	0.6042	+0.0112	+ 1.9
9.0	0.6374	0.6330	-0.0044	- 0.69

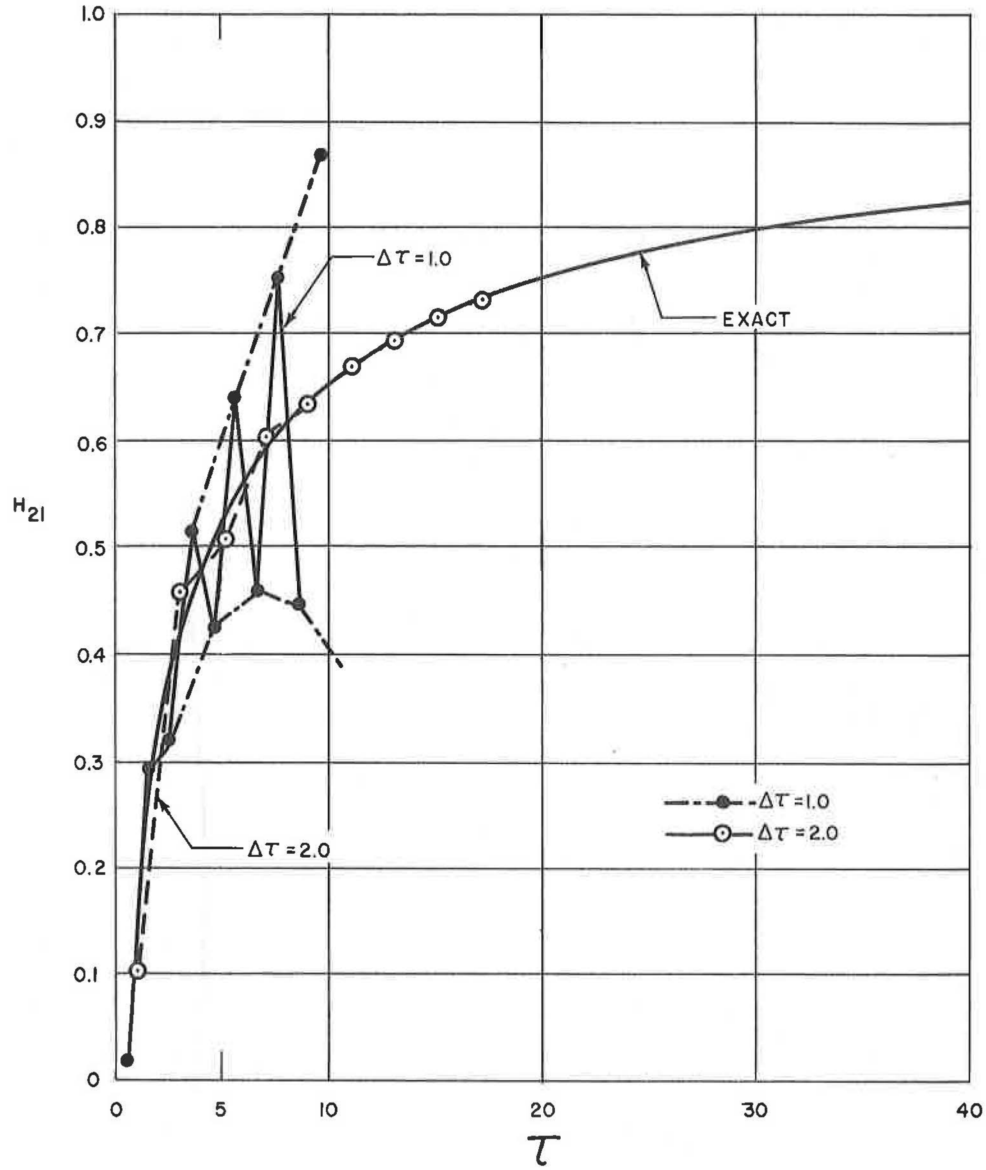

Figure 3 GRAPH OF RESULTS OF CALCULATION FOR H_{21} USING STOLZ METHOD
61-1203

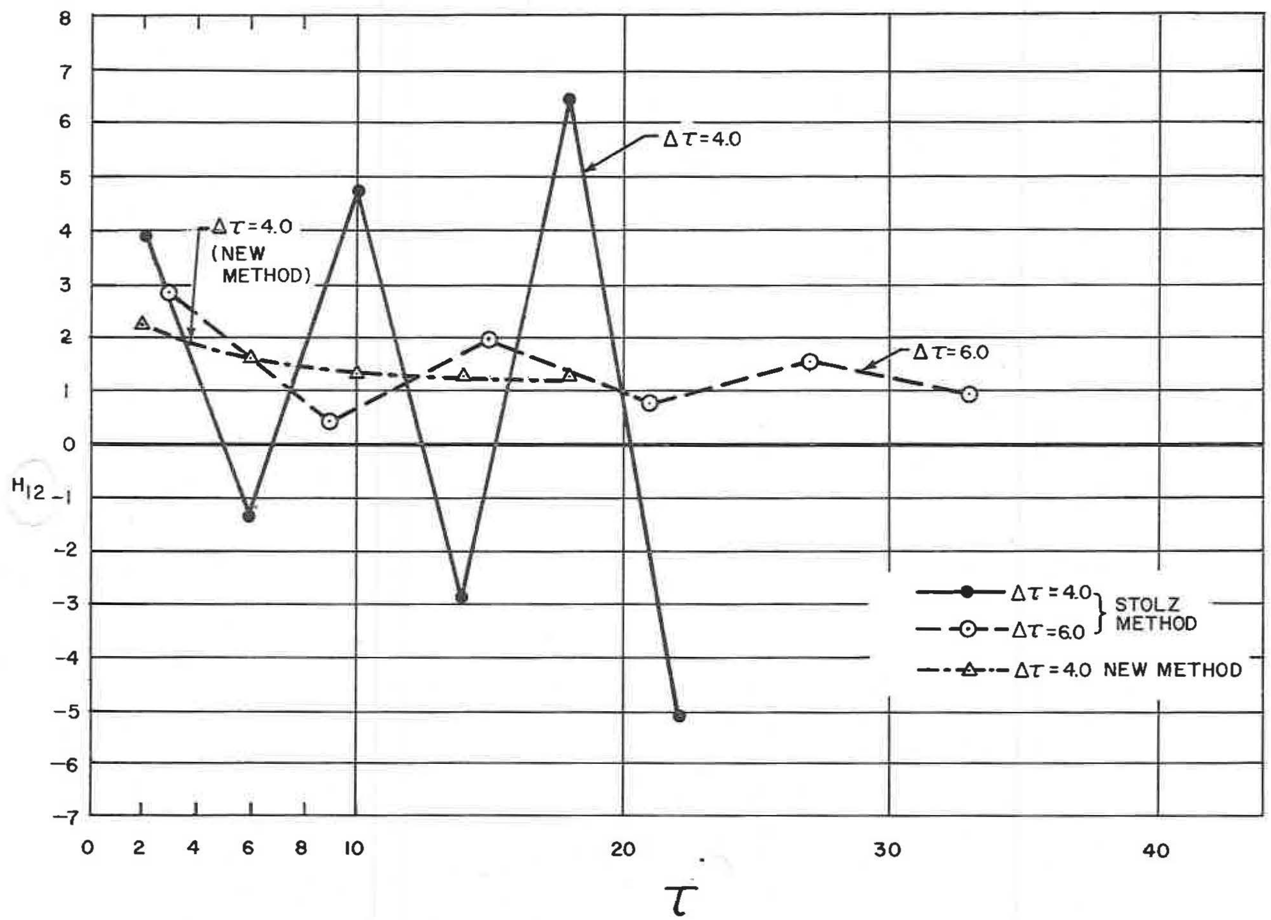

Figure 4 GRAPH OF RESULTS OF CALCULATION FOR H_{12} USING STOLZ METHOD

61-1204

The calculation for H_{12} is more difficult as shown by figure 4 since a stable solution is not obtained until $\Delta\tau = 6$. Values for H_{12} for $\Delta\tau = 1, 2, 4$, and 6 are given by Table II. It may be noted that the oscillations become much larger as the time step is reduced.

B. DISCUSSION OF DAMPING IN SYSTEM

In the calculations for both H_{21} and H_{12} instability is obtained when the calculation time step is reduced to sufficiently small values. This is caused in part by the temperature lag at an interior point after a change of heat flux at the surface. For example, consider the partial derivatives of θ_1 and θ_2 with respect to time which are shown by figure 5. $\partial\theta_1/\partial t$ is used in equation (4a) to calculate H_{21} and $\partial\theta_2/\partial t$ in equation (4b) for H_{12}. In both derivatives there is a time delay δ before the derivatives begin to increase measurably in value. For θ_1, it is about $\tau = 0.25$ and for θ_2 the lag δ is about $\tau = 1$. Then almost no information regarding the surface heat flux from time $t = 0$ to δ is received at the interior point at time δ. Consequently, it is very difficult to determine exactly the surface heat flux at a time t from a knowledge of all temperatures before time t but none after time t.

This result may be stated mathematically. Equation (1) may be re-written as

$$T(x_1,t) = \int_{\delta}^{t} \dot{q}(0,t-\lambda)\frac{\partial\phi(x_1,t)}{\partial t}d\lambda + \int_{0}^{\delta} \dot{q}(0,t-\lambda)\frac{\partial\phi(x_1,t)}{\partial t}d\lambda \quad . \tag{5}$$

TABLE II

RESULTS OF CALCULATION FOR H_{12} USING PROBLEM 793.02 DIGITAL PROGRAM (STOLZ METHOD)

$\Delta\tau = 1.0$		$\Delta\tau = 2.0$		$\Delta\tau = 4.0$		$\Delta\tau = 6.0$	
τ	H_{12}	τ	H_{12}	τ	H_{12}	τ	H_{12}
0.5	51.8	1.0	9.8	2.0	3.97	3.0	2.84
1.5	-683.3	3.0	-24.7	6.0	-1.33	9.0	0.44
2.5	9,680.5	5.0	91.8	10.0	4.74	15.0	1.94
3.5	-136,944.5	7.0	-310.2	14.0	-2.89	21.0	0.77

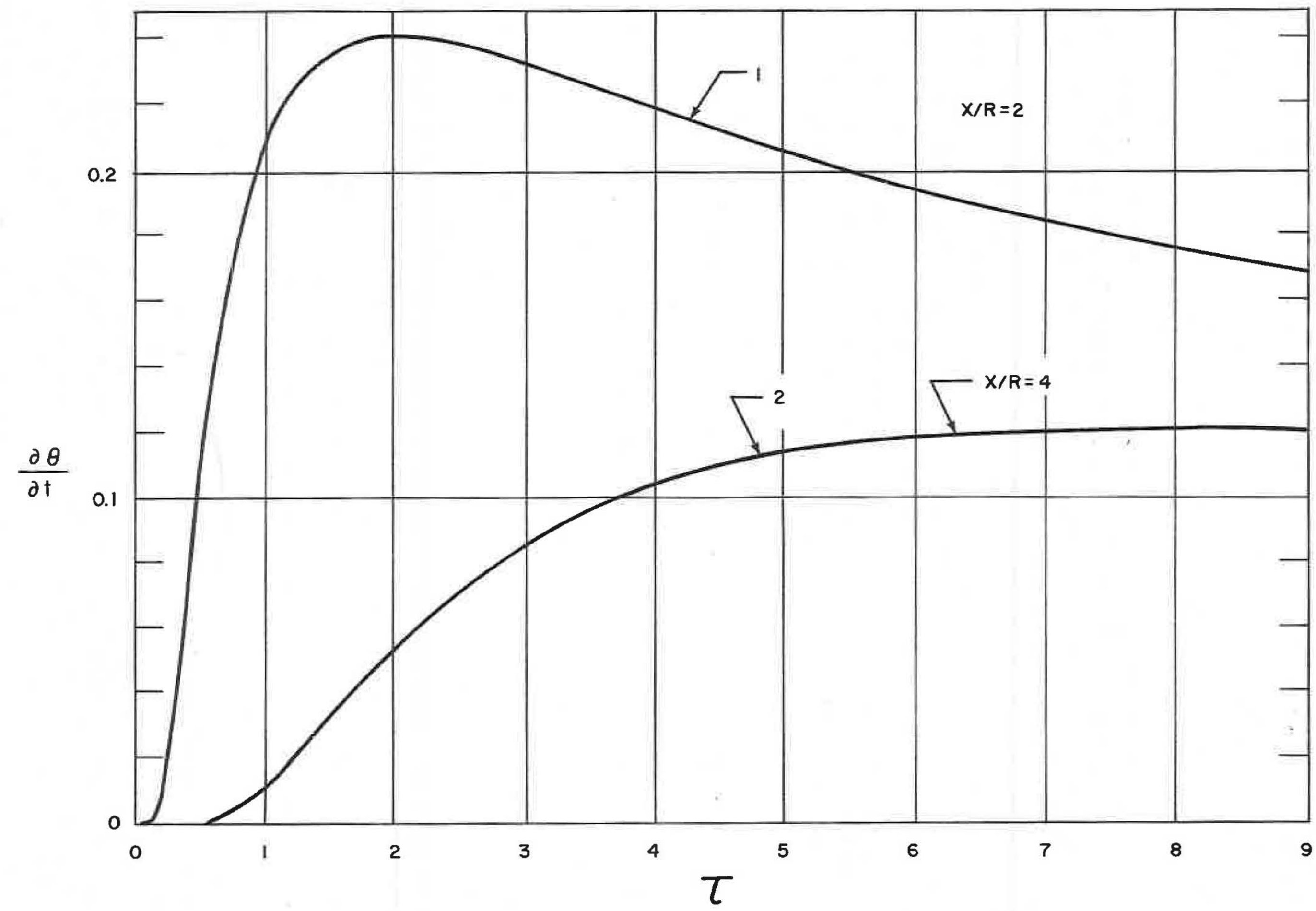

Figure 5 TIME DERIVATIVE $\partial\theta / \partial t$ FOR POINTS 2R and 4R BELOW SURFACE HEATED WITH UNIT STEP HEAT FLUX
61-1205

Now the second integral in equation (5) is essentially zero since $\partial\theta/\partial t$ is almost zero from $t = 0$ to δ. But the $\dot{q}$'s between times $t - \delta$ and t are involved in only this integral. The conclusion is then reached that $\dot{q}$ cannot be readily determined at any time between $t - \delta$ and t by using the temperature T only until time t. In any improved numerical technique to calculate the inverse convolution the temperatures at times greater than t should be used to calculate $\dot{q}$ at time t. This is done in a new numerical procedure which is to be described.

The time lag in the temperature response is a result of damping in the system. This damping results in this case from the behavior of the governing differential equation of transient heat conduction,

$$\alpha \nabla^2 T = \frac{\partial T}{\partial t} \quad . \tag{6}$$

Another example of this damping may be obtained by considering the solution to the conduction problem when a surface heat flux $\dot{q}$ is caused to vary in time as $\dot{q}_\epsilon \cos(\omega t)$. The steady state temperature response ϵ for this flux is given by[10]

$$\epsilon(x,t) = \frac{\dot{q}_\epsilon}{k}\left(\frac{\alpha}{\omega}\right)^{1/2}\left\{\cos\left[\omega t - x\left(\frac{\omega}{2\alpha}\right)^{1/2} - \frac{\pi}{4}\right]\right\} e^{-x\left(\frac{\omega}{2\alpha}\right)^{1/2}} \quad . \tag{7}$$

This equation indicates that the temperature response at an interior point x due to a large value of $\dot{q}_\epsilon$ might be made as small as desired by increasing the value of ω appropriately. This observation may be applied to the calculation of the inverse convolution. Adding ϵ to equation (1) gives

$$T(x_1,t) + \epsilon(x_1,t) = \int_0^t \left[\dot{q}(x_0,\lambda) + \dot{q}_\epsilon(x_0)\cos(\omega\lambda)\right] \frac{\partial \phi(x_1, t-\lambda)}{\partial t} d\lambda \quad . \tag{8}$$

Let ϵ be small relative to T. Now this does not mean that $\dot{q}_\epsilon$ is small relative to $\dot{q}$ since ϵ may be made small by letting ω become large. It may be concluded then, that, due to damping in the system, a sinusoidal variation of heat flux at the surface cannot be detected physically at an interior point if ω is too large. ω will be interpreted to be "too large" in the numerical calculations of the inverse convolution when $\dot{q}$ oscillates from time step to time step. This restriction is to be made in the new numerical procedure since the procedure cannot yield accurate information about sinusoidal fluxes of higher frequencies.

C. NEW INVERSE CONVOLUTION PROCEDURE

An inverse convolution procedure which incorporates the features mentioned above will now be given.* The procedure is related to the method of least squares of solving integral equations which is discussed by Hildebrand.[5] This latter method was suggested by Stolz[2] in response to questions regarding an alternate formulation of the problem. This method, however, is given for a Fredholm rather than a Volterra integral equation.

Equation (2a) which is

$$T_M = \sum_{n=1}^{M-1} \dot{q}_n \Delta\phi_{M-n} + \dot{q}_M \Delta\phi_0 \quad , \tag{2a}$$

will be used along with an equation for time M + 1,

$$T_{M+1} = \sum_{n=1}^{M-1} \dot{q}_n \Delta\phi_{M-n+1} + \dot{q}_M \Delta\phi_1 + \dot{q}_{M+1} \Delta\phi_0 \quad . \tag{9}$$

T_M and T_{M+1} will not be the given temperatures at times $M\Delta t$ and $(M+1)\Delta t$ as in program 793.02. Instead these will be approximately the temperatures at those times. This is because a small perturbation ϵ in one of these temperatures might cause a large variation in flux as discussed above. T_M and T_{M+1} will be related to the experimentally measured temperatures ${}_eT_M$ and ${}_eT_{M+1}$ by a least squares relation. The condition is to be when

$$\left(T_M - {}_eT_M\right)^2 + \left(T_{M+1} - {}_eT_{M+1}\right)^2 = F \quad , \tag{10}$$

is equal to a minimum.

There are three equations (Equations (2a), (9), and (10)) from which to calculate $\dot{q}_M, \dot{q}_{M+1}$, T_M, and T_{M+1}. One further relation is then needed.

* The procedure is only one of a family which might be used. It appears to be the simplest and is the one which has been primarily investigated. Other procedures will be suggested in the next section.

Some assumption relative to $\dot{q}_{M+1}$ will now be made. $\dot{q}_{M+1}$ is to be made equal* to $\dot{q}_M$ in equation (9) for the calculation of $\dot{q}_M$. This assumption is validated by the experimental results which will be described. One fact that suggests that the assumption may be made is that

$$\dot{q}_{M+1} \Delta\phi_0 < \dot{q}_M \Delta\phi_1 \quad . \tag{11}$$

To find the minimum value of F in equation (10), F is differentiated with respect to $\dot{q}_M$ and set equal to zero to obtain

$$\left(T_M - {}_eT_M\right)\frac{dT_M}{d\dot{q}_M} + \left(T_{M+1} - {}_eT_{M+1}\right)\frac{dT_{M+1}}{d\dot{q}_M} = 0 \tag{12}$$

Upon introducing the expressions for T_M and T_{M+1} from equations (2a) and (9) into equation (12) and simplifying one may obtain

$$\dot{q}_M = \frac{1}{C_2}\left[{}_eT_M + C_1\,{}_eT_{M+1} - \sum_{n=1}^{M-1} \dot{q}_n\,\bar{\phi}_{M-n}\right] , \tag{13a}$$

where

$$C_1 = 1 + \frac{\Delta\phi_1}{\Delta\phi_0} , \tag{13b}$$

$$C_2 = \Delta\phi_0\left[1 + C_1^{\,2}\right] , \tag{13c}$$

$$\bar{\phi}_{M-n} = \Delta\phi_{M-n} + C_1\,\Delta\phi_{M-n+1} \quad . \tag{13d}$$

This is the new recurrence expression for calculating the inverse convolution. Equation (13) has been programmed and is called by the Mathematics Section Problem 895.02. Hyman Shumrak was the programmer. The input required for this program includes a table of ${}_eT$ versus time t and a table of ϕ versus time t . For values of both ${}_eT$ and ϕ which are required between tabulated values of t , the program uses a least squares parabola passed through four adjacent points. The required values are taken between the two mid-points

* This assumption was found experimentally to yield more stable results than if $\dot{q}$ had been assumed to be linear from time $(M-1)\Delta t$ to $(M+1)\Delta t$.

TABLE III

RESULTS OF CALCULATION FOR H_{21} USING
PROBLEM 895.02 DIGITAL PROGRAM (NEW METHOD)

$\Delta\tau = 0.5$				
τ	H_{21} (Exact)	H_{21}	$H_{21} - H_{21}(ex)$	$\frac{H_{21} - H_{21}(ex)}{H_{21}(ex)}$ %
0.25	0.0047	0.0280	+0.0233	497
0.75	0.1025	0.0977	-0.0048	-4.9
1.25	0.2059	0.1972	-0.0087	-4.4
1.75	0.2851	0.2840	-0.0011	-0.38
2.25	0.3458	0.3454	-0.0004	-0.12

$\Delta\tau = 1.0$				
τ	H_{21} (Exact)	H_{21}	$H_{21} - H_{21}(ex)$	$\frac{H_{21} - H_{21}(ex)}{H_{21}(ex)}$ %
0.5	0.0455	0.0966	+0.0511	112
1.5	0.2482	0.2364	-0.0118	-4.7
2.5	0.3711	0.3607	-0.0104	-2.8
3.5	0.4497	0.4448	-0.0049	-1.1
4.5	0.5050	0.5021	-0.0029	-0.58

$\Delta\tau + 2.0$				
τ	H_{21} (Exact)	H_{21}	$H_{21} - H_{21}(ex)$	$\frac{H_{21} - H_{21}(ex)}{H_{21}(ex)}$ %
1.0	0.1573	0.2294	0.0721	+4.6
3.0	0.4142	0.3951	-0.0191	-4.6
5.0	0.5271	0.5114	-0.0157	-3.0
7.0	0.5930	0.5840	-0.0090	-1.5
9.0	0.6374	0.6318	-0.0056	-0.88

except for values required before the first two and after the last two tabulated values of ${}_eT$ and ϕ . In the latter cases the values are simply read from the first and last least squares curves respectively. The slight amount of smoothing introduced by the least squares curves was found to be desirable since the convolution involves a derivative which is quite sensitive to roughness in input data.

Comparisons of the improved method, Problem 895.02, with Problem 793.02 will now be given. The same examples as used above for the latter program will be used in for the comparisons. H_{21} , the "easy" kernel to calculate, will be considered first. Calculations have been made for $\Delta\tau$ = 0.5, 1.0, and 2.0. These results are shown by figure 6. This curve is to be compared with figure 3. The new method yields non-oscillatory results in contrast with the present method for $\Delta\tau$ as small as $\Delta\tau$ = 0.5. Perhaps even more impressive is the fact that the $\Delta\tau$ = 1 result shown by figure 3 is unstable while for the smaller time step result of $\Delta\tau$ = 0.5 shown by figure 4 is quite accurate after the first time step. For time steps about $\Delta\tau$ = 0.25 the new method also begins to yield oscillatory results; for even smaller time steps it will become unstable.

Table III, which is to be compared with Table I, gives the results for H_{21} for the $\Delta\tau$ = 0.5, 1, and 2 cases. In each case, the first time step gave H_{21} values which are too large. It may be noted for the first time step that the value of the error, $H_{21} - H_{21(Exact)}$, decreases as the time step interval is decreased; the percent error increases however. In each case the error decreases with time as was also noted in Table I for the $\Delta\tau$ = 2 case. For this example, the results for the new inverse convolution calculation procedure converge to the correct values as $\Delta\tau$ is decreased until $\Delta\tau$ = 0.5.

The more difficult kernel to calculate in the examples given for Problem 793.02 was H_{12} . It is found also to be more difficult using Problem 895.02. Results for $\Delta\tau$ = 1 and 2 calculations for H_{12} are shown by figure 7. Both these curves appear to possess oscillations with those for $\Delta\tau$ = 1 being the larger. Calculations for $\Delta\tau$ = 0.5 are unstable and possess violent oscillations. These results should be compared with the results shown by figure 4; in the latter case the $\Delta\tau$ = 4 results were unstable. The $\Delta\tau$ = 4 results using the improved method are also shown by figure 4 for comparison. Consequently for this example the improved procedure permits calculations with time steps about one-eighth those of the older procedure.

The point has been made that H_{12} does not exist in a strict mathematical sense. Yet calculations for H_{12} appear to converge for τ greater than 10 while using time steps larger than $\Delta\tau$ = 1 (see figure 8). H_{12} may be incorrectly found to be (see Ref. 9)

$$H_{12} = 2 - \operatorname{erfc} \tau^{-1/2} . \tag{14}$$

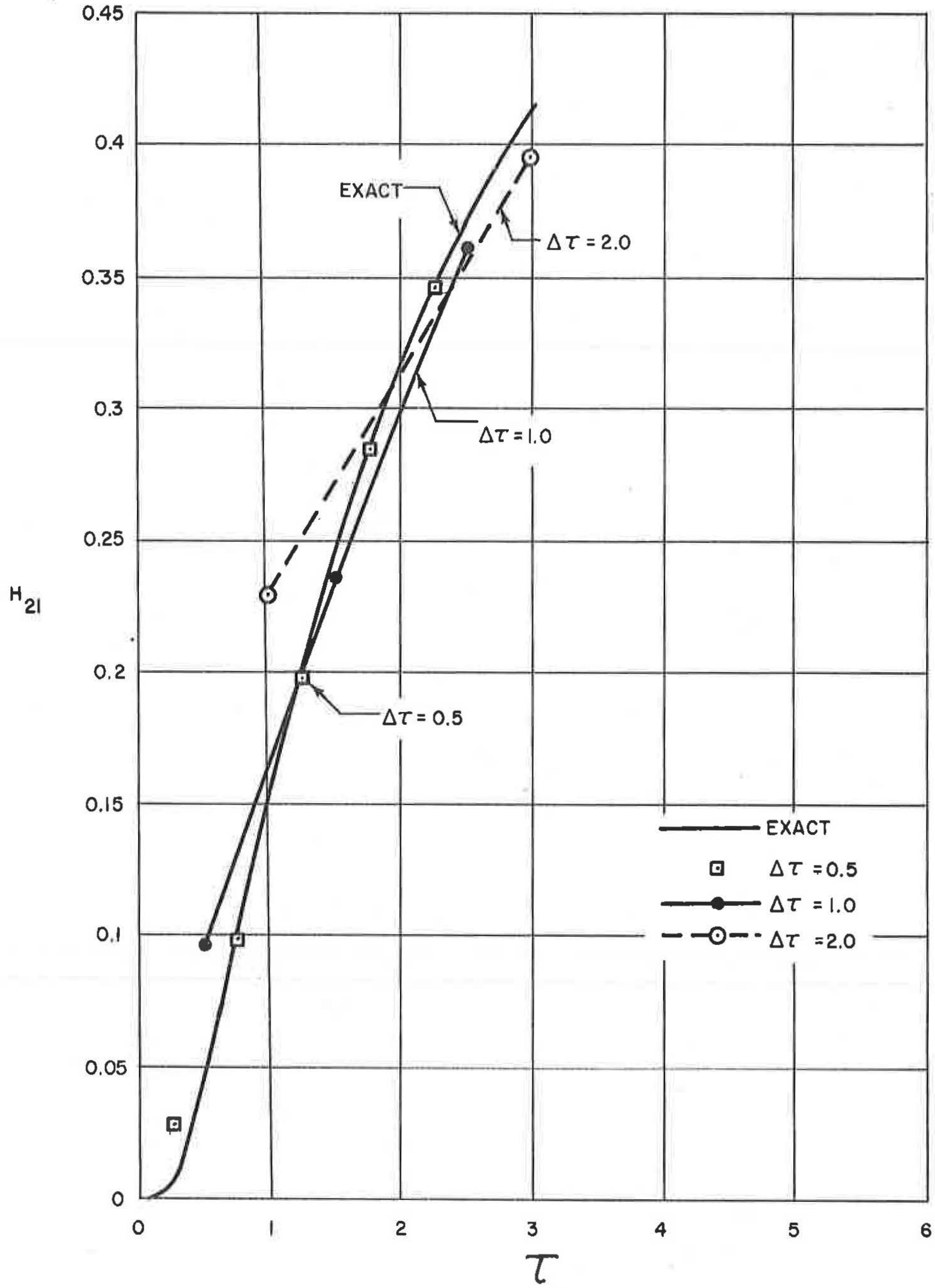

Figure 6 GRAPH OF RESULTS OF CALCULATION FOR H_{21} USING NEW METHOD
61-1206

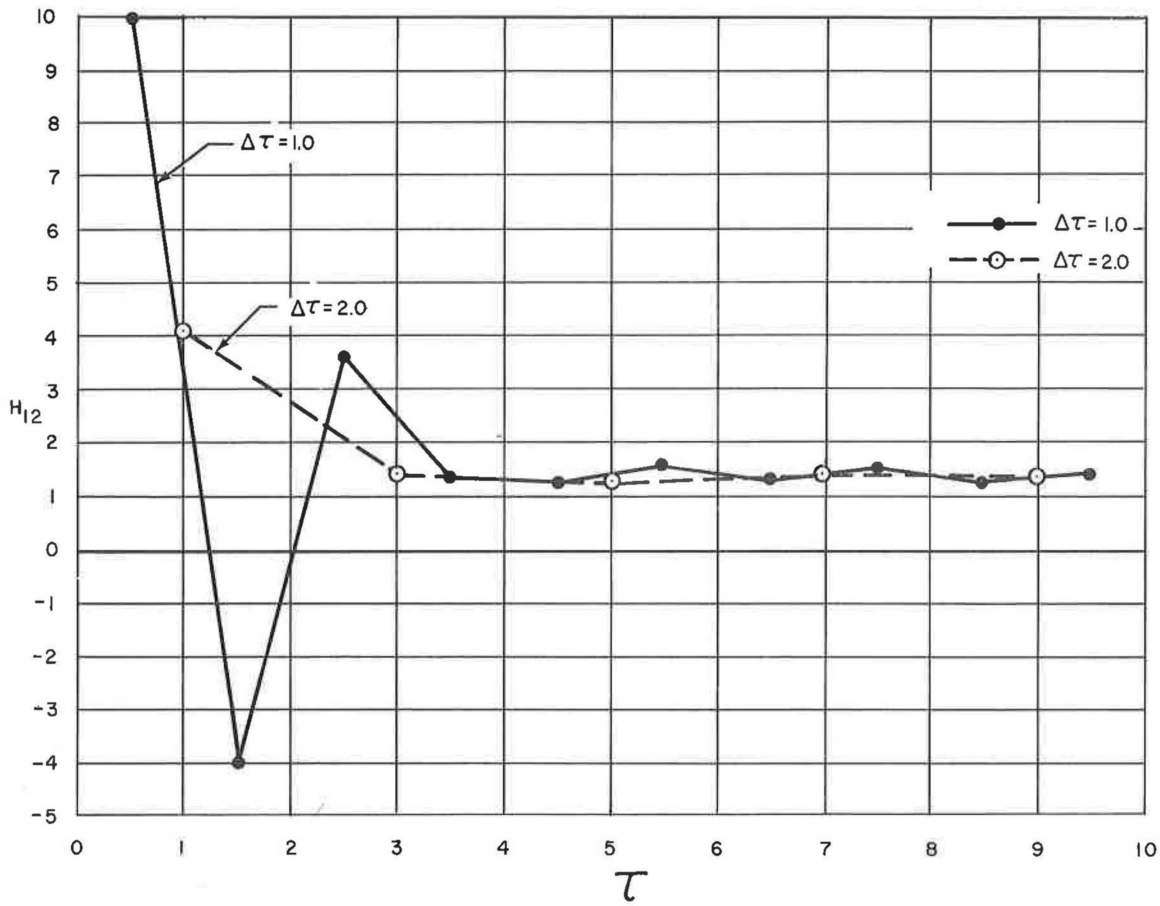

Figure 7 GRAPH OF RESULTS OF CALCULATION FOR H_{12} USING NEW METHOD
61-1207

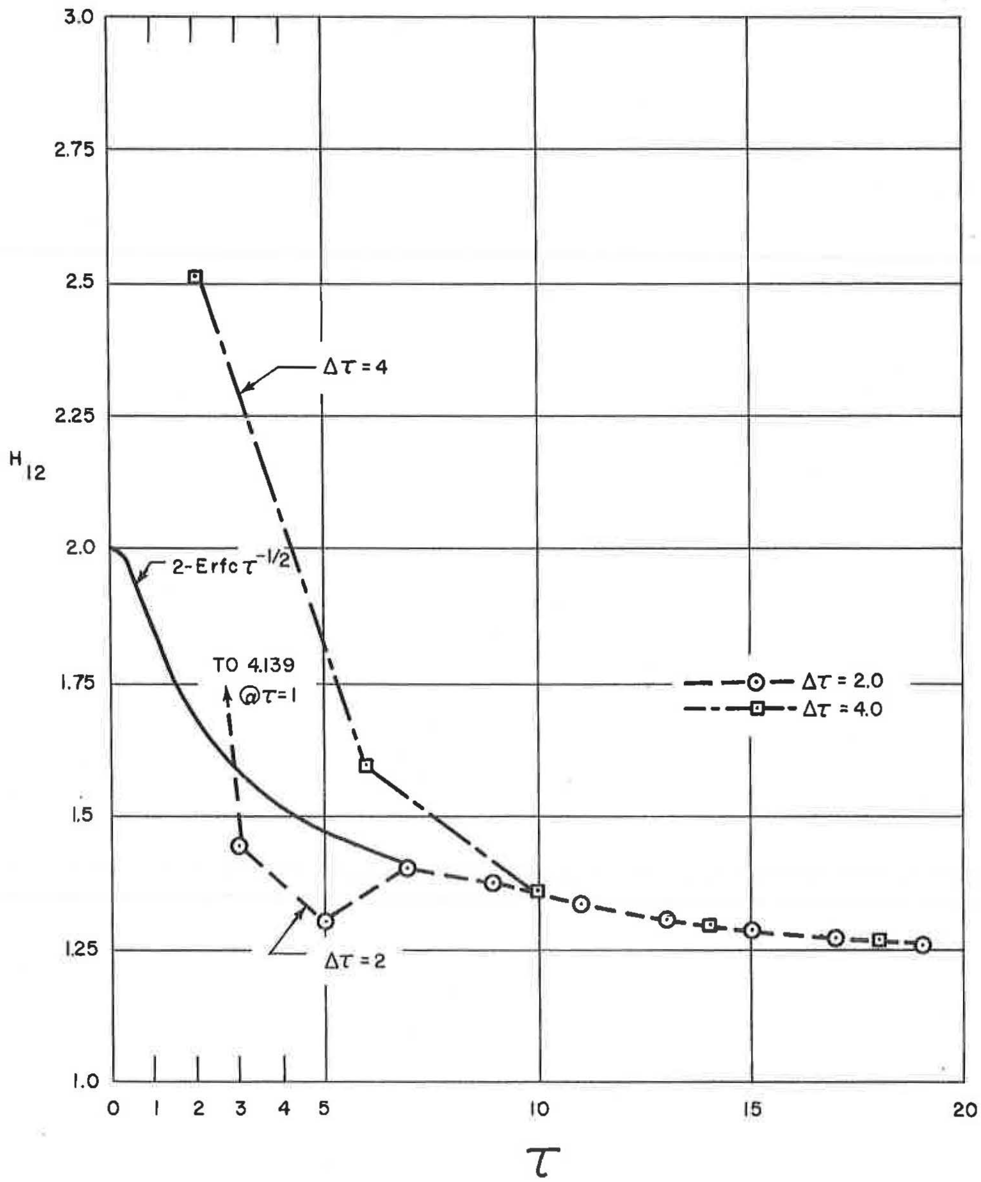

Figure 8 COMPARISON OF Erfc $\tau^{-1/2}$ AND H_{12} CALCULATED USING NEW METHOD
61-1208

This equation is shown also by figure 8. After $\tau = 7$ for $\Delta\tau = 2$ and after $\tau = 10$ for $\Delta\tau = 4$ the values of H_{12} calculated using the inverse convolution are almost identical to the results obtained from equation (14). In fact, at $\tau = 9$ for $\Delta\tau = 2$ the difference is only + 0.03 percent. The difficulty in calculating H_{12} is apparently caused by the fact that it does not exist near $\tau = 0$.

It has been demonstrated for one case that the new inverse convolution equation (Eq. 13) can be used to calculate the surface heat flux from an interior measurement. One might now inquire if the equation could be used with comparable success for another case. The heat flux or kernel function H_{21} which was calculated is a "good" function in that all its derivatives are zero at $\tau = 0$. A more rigorous test of the new equation would be to consider a case in which the heat flux starts at infinity and then rapidly decreases in magnitude. Such a case may be readily devised.

The problem in which the surface temperature of a heat conducting solid is suddenly increased to a constant value is a convenient one for our purposes. (See Appendix B.) The surface heat flux in this case varies at $\tau^{-1/2}$ and hence is infinite at $\tau = 0$ and zero at τ = infinity. A calculation for this heat flux was made using Problem 895.02. The transient temperature at 2R below the heated surface resulting from this time-variable flux and given by equation B-2 was used as input along with the temperature at 2R given by equation A-3 caused by a constant surface heat flux. Some of the results of the calculations are shown by figure 9. The calculation for $\Delta\tau = 0.5$ is stable and gives reasonably accurate values though a slight oscillation (which does not grow) is noted. For $\Delta\tau = 1.0$ the results are initially less accurate but no oscillations are noted. As the calculation step is made smaller than $\Delta\tau = 0.5$ more oscillations are observed and finally the calculations become unstable. This latter result was also found while calculating H_{21}. This observation is quite important since it appears to indicate the following. While predicting heat fluxes at a surface from an interior temperature with Problem 895.02, the minimum calculation time Δt_{min} should be chosen to be

$$\frac{\alpha \Delta t_{min}}{R^2} \geq \frac{1}{2} , \tag{15}$$

when the temperature is given at a point 2R below the surface. It would be more convenient to let in equation (15)

$$2R = \delta , \tag{16}$$

and then equation (15) becomes

$$\frac{\alpha \Delta t_{min}}{\delta^2} \geq \frac{1}{8} \tag{17}$$

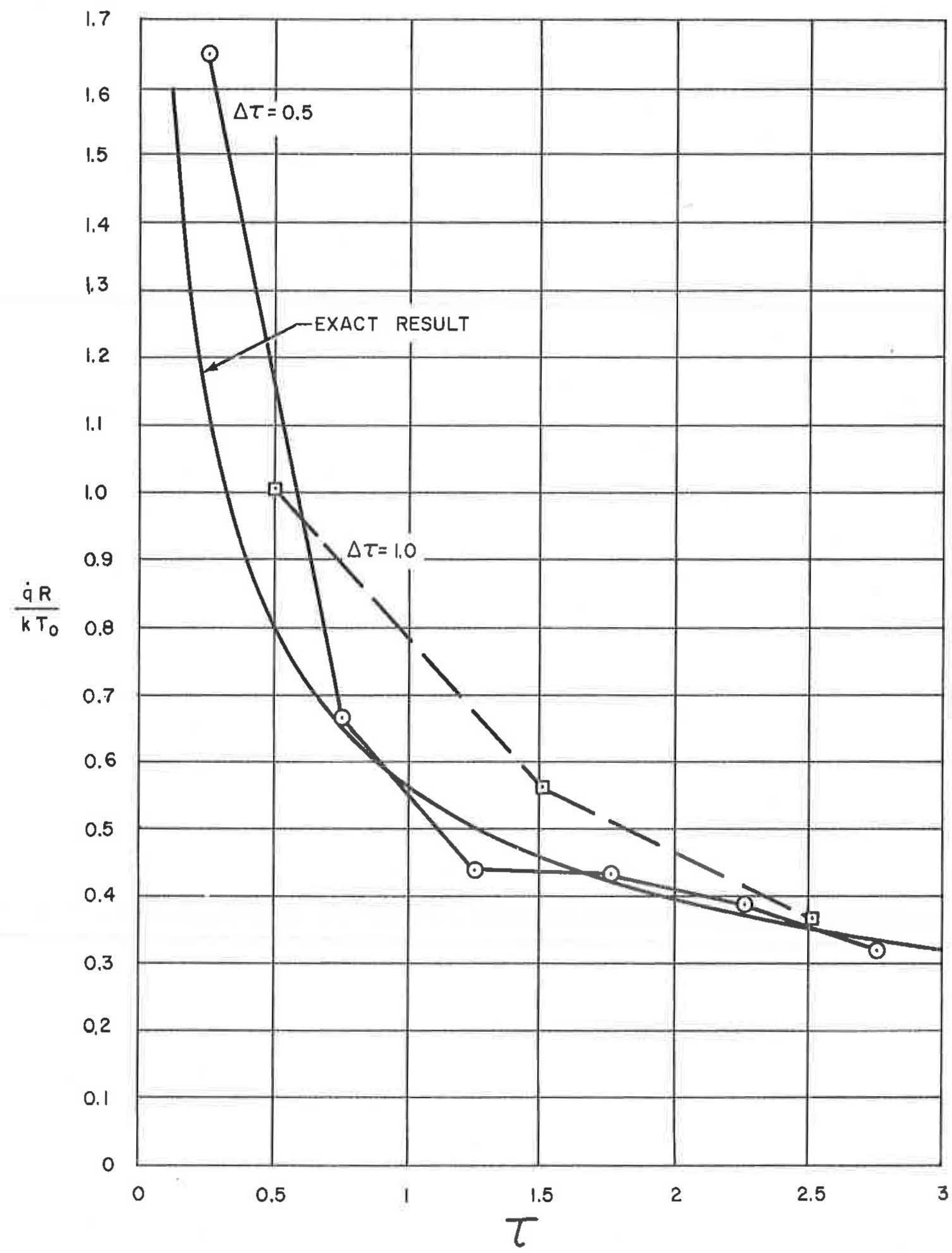

Figure 9 RESULTS OF CALCULATION FOR SURFACE FLUX WHICH VARIES AS $\tau^{-1/2}$
61-1209

Thus when the temperature is given δ below the heated surface in a one-dimensional body, the minimum calculation time should be $8\ \delta^2/\alpha$.

The stability of equation (13) should be investigated analytically rather than solely experimentally. A stability analysis has been begun by the author but it has not been completed at the writing of this paper. Until this analysis is completed the criterion of equation (17) is recommended.

D. ALTERNATE METHODS TO COMPUTE INVERSE CONVOLUTION

The equation which has been used to calculate the inverse convolution (Eq. 13) is only one of many which could be developed. Others might use Simpson's rule or some other integration weighting functions for example. In addition, more than one future temperature could be used in calculating the inverse convolution.

A way in which equation (13) may be extended to use more than one future temperature will now be given. The method will be illustrated by using two future temperatures. The starting equations for this case are

$$T_M = \Sigma_0 + \dot{q}_M \, \Delta\phi_0 \ , \tag{18a}$$

$$T_{M+1} = \Sigma_1 + \dot{q}_M \, \Delta\phi_1 + \dot{q}_{M+1} \, \Delta\phi_0 \ , \tag{18b}$$

$$T_{M+2} = \Sigma_2 + \dot{q}_M \, \Delta\phi_2 + \dot{q}_{M+1} \, \Delta\phi_1 + \dot{q}_{M+2} \, \Delta\phi_0 \ , \tag{18c}$$

where

$$\Sigma_x = \sum_{n=1}^{M-1} \dot{q}_n \, \Delta\phi_{M-n+x}$$

$$\left(T_M - {}_eT_M\right)^2 + \left(T_{M+1} - {}_eT_{M+1}\right)^2 + \left(T_{M+2} - {}_eT_{M+2}\right)^2 = \text{minimum} \ . \tag{18d}$$

1. Constant Future Heat Fluxes Assumption

In satifying equations (18a), (18b), and (18c) in a "least squares sense" $\dot{q}$ might be assumed to be

$$\dot{q}(t) = A + B\left(\frac{t}{\Delta t} - M\right) + C\left(\frac{t}{\Delta t} - M\right)^2 . \tag{19}$$

If, however, the three constants A, B, and C in equation (19) are used with the three equations (equations (18a), (18b), and (18c)) the temperatures will be satisfied exactly. For this reason if an averaging process is to be incorporated then there must be more equations of the form of equation (18a) than constants to be determined in equation (19). Consequently, C will be assumed to be equal to zero. If B is also made equal to zero, then the q's in equation (18) are all equal,

$$\dot{q}_M = \dot{q}_{M+1} = \dot{q}_{M+2} . \tag{20a}$$

Equations (18a), (18b) and (18c) may then be written

$$T_M = \Sigma_0 + a_0 \dot{q}_M , \tag{20b}$$

$$T_{M+1} = \Sigma_1 + a_1 \dot{q}_M , \tag{20c}$$

$$T_{M+2} = \Sigma_2 + a_2 \dot{q}_M , \tag{20d}$$

where

$$a_i = \sum_{j=0}^{i} \Delta\phi_j = \phi_{i+1} - \phi_0 ,$$

or

$$a_0 = \Delta\phi_0 = \phi_1 - \phi_0 ; \; a_1 = \Delta\phi_0 + \Delta\phi_1 = \phi_2 - \phi_0 , \text{ etc.}$$

In this case, equation (18d) will be differentiated only with respect to $\dot{q}_M$ to obtain

$$\left(T_M - {}_eT_M\right) a_0 + \left(T_{M+1} - {}_eT_{M+1}\right) a_1 + \left(T_{M+2} - {}_eT_{M+2}\right) a_2 = 0 . \tag{20e}$$

Solving equations (20b) to (20e) for $\dot{q}_M$ gives

$$\dot{q}_M = \frac{\left(a_0 e_{T_M} + a_1 e_{T_{M+1}} + a_2 e_{T_{M+2}}\right) - \sum_{n=1}^{M-1} \dot{q}_n \left[a_0 \Delta\phi_{M-n} + a_1 \Delta\phi_{M-n+1} + a_2 \Delta\phi_{M-n+2}\right]}{a_0^2 + a_1^2 + a_2^2} . \qquad (20f)$$

It may seen from equation (20f) how to extend it to treat "r" future times instead of only 2 as used in equation (20f); the result is

$$\dot{q}_M = \frac{\sum_{i=0}^{r} a_i e_{T_{M+i}} - \sum_{n=1}^{M-1} \dot{q}_n \Phi_{M-n}}{\sum_{i=0}^{r} a_i^2} , \qquad (20g)$$

where

$$\Phi_{M-n} = \sum_{i=0}^{r} a_i \Delta\phi_{M-n+i} .$$

Equation (20g) is a general recurrence relation to calculate $\dot{q}_M$ assuming that "r" future $\dot{q}$'s are equal to $\dot{q}_M$. When $r = 1$, equation (20g) will reduce to equation (13a).

2. <u>Linear Future Heat Fluxes Assumption</u>

When only "C" in equation (19) is made equal to zero, the $\dot{q}$'s become

$$\dot{q}_M = A , \qquad (21a)$$

$$\dot{q}_{M+1} = A + B = \dot{q}_M + B , \qquad (21b)$$

$$\dot{q}_{M+2} = A + 2B = \dot{q}_M + 2B \,. \tag{21c}$$

These $\dot{q}$'s are then assumed to be linear with respect to time only for the calculation of $\dot{q}_M$. These relations are now introduced into equations (18a), (18b), and (18c) to obtain

$$T_M = \Sigma_0 + a_0 \dot{q}_M \,, \tag{22a}$$

$$T_{M+1} = \Sigma_1 + a_1 \dot{q}_M + b_0 B \,, \tag{22b}$$

$$T_{M+2} = \Sigma_2 + a_2 \dot{q}_M + b_1 B \,, \tag{22c}$$

where

$$b_\ell = \sum_{i=0}^{\ell} a_i = \sum_{i=0}^{\ell} \phi_{i+1} - (\ell + 1)\phi_0 \,,$$

or

$$b_0 = a_0 = \phi_1 - \phi_0$$

$$b_1 = a_0 + a_1 = \phi_2 \; (+\phi_1) - 2\phi_0 \,, \text{ etc.} \qquad b_2 = \phi_1 + \phi_2 + \phi_3 - 3\phi_0$$

Equations (22a), (22b), and (22c) are now inserted into equation (18d). The partial derivative of equation (18d) is taken with respect to $\dot{q}_M$ and B. The former result is given by equation (20e) and the latter by

$$\left(T_{M+1} - {}_eT_{M+1}\right) b_0 + \left(T_{M+2} - {}_eT_{M+2}\right) b_1 = 0 \,. \tag{22d}$$

Combining equations (20e) and (22a) to (22d) and solving for $\dot{q}_M$ gives

$$q_M = \frac{\left[a_0\, e_{T_M} + (a_1 - b_0 E_2)\, e_{T_{M+1}} + (a_2 - b_1 E_2)\, e_{T_{M+2}}\right]}{D_2}$$

$$- \frac{\sum_{n=1}^{M-1} \dot{q}_n \left[a_0\, \Delta\phi_{M-n} + (a_1 - b_0 E_2)\, \Delta\phi_{M-n+1} + (a_2 - b_1 E_2)\, \Delta\phi_{M-n+2}\right]}{D_2} \quad (22e)$$

where

$$D_2 = a_0^2 + a_1^2 + a_2^2 - E_2\,(a_1 b_0 + a_2 b_1)\ ,$$

$$E_2 = \frac{a_1 b_0 + a_2 b_1}{b_0^2 + b_1^2}\ .$$

Equations (20f) and (20e) might now be compared. The former assumes that the subsequent $\dot{q}$'s are constant and the latter that the $\dot{q}$'s are linear. Equation (22e) reduces to equation (20f) if E is made equal to zero. Since the difference between these two equations is only certain constants which may be found once at the beginning of the $\dot{q}$ calculation, the digital calculation time would be about equal for the same time steps.

The linear approximation for $\dot{q}$ equation, equation (22e) might be extended to include "r" future times. The equation would be

$$q_M = \frac{\sum_{i=0}^{r} \bar{a}_i\, e_{T_{M+i}} - \sum_{n=1}^{M-1} \dot{q}_n\, \bar{\Phi}_{M-n}}{D_r}\ , \quad (22f)$$

where

$$\bar{a}_i = a_i - b_{i-1} E_r\ ,$$

$$b_{-1} \equiv 0\ ,$$

$r = 3$

$$E_r = \frac{\sum_{i=1}^{r} a_i b_{i-1}}{\sum_{i=1}^{r} b_{i-1}^2}, \qquad E_3 = \frac{a_1 b_0 + a_2 b_1 + a_3 b_2}{b_0^2 + b_1^2 + b_2^2}$$

$$D_r = \sum_{i=0}^{r} a_i^2 - E_r \sum_{i=1}^{r} a_i b_{i-1}, = a_0^2 + a_1^2 + a_2^2 + a_3^2 - E_3 (a_1 b_0 + a_2 b_1 + a_3 b_2)$$

$\bar{a}_i = a_i - b_{i-1} E_r$

$$\bar{\Phi}_{M-n} = \sum_{i=0}^{r} \bar{a}_i \, \Delta\phi_{M-n+i} \cdot (a_0)\Delta\phi_{M-n} + (a_1 - b_0 E_3)\Delta\phi_{M-n+1} + (a_2 - b_1 E_3)\Delta\phi_{M-n+2} + (a_3 - b_2 E_3)\Delta\phi_{M-n+3}$$

E. FURTHER USES OF THE INVERSE CONVOLUTION

The inverse convolution may be used to calculate the heat flux at a surface from an exact interior temperature. A related use is the calculation of the surface temperature from an exact interior temperature. It is shown in reference 1 that the surface heat flux and temperature may be also calculated (by using the inverse convolution) from an interior temperature measurement. The temperature measurement may be in error due to the presence of the thermocouple itself.

The inverse convolution may be used to calculate a kernel function in addition to a heat flux. This use has also been suggested above; it is described at some length in reference 1. It has been used to calculate a number of kernel functions for the case when a thermocouple is embedded in a low conductivity material. The paper reporting these results is to be Part III of this report.[11]

Another possible use of the inverse convolution is as a tool to aid in the solution of certain heat conduction problems. The concept is to employ the inverse convolution to add certain heat conduction solutions to obtain further solutions of more difficult problems. This may be explained more clearly by using an example.

A finite heat sink is to be heated uniformly with a constant heat flux on one surface and insulated on the other. The insulated surface temperature is to be measured by a thermocouple placed normal to this surface (Fig. 10). The thermocouple material T will be assumed to have a high thermal conductivity relative to the heat sink material S. The problem now is to determine the disturbed temperature at point "p" and to find the temperature disturbance $T_p - T_{p\infty}$.

In solving this problem the solutions to three auxiliary problems are needed. The geometries and boundary conditions for these three problems are shown by figures 11A, 11B, and 11C. The first problem (Fig. 11A) is the determination of temperature at the insulated surface of a one-dimensional heat sink. This solution is well known and is given by Carslaw and Jaeger.[10] This temperature will be denoted $T_{p\infty}$ and is the undisturbed temperature in the heat sink.

In figure 11B a constant heat flux $\dot{q} = -1$ is applied over the cylindrical area from $r = 0$ to R at $z = L$. The other surface areas are insulated. Since the temperature over this heated area varies somewhat with position[12], the mean temperature over this area will be used; it will be designated ϕ_B. The temperature at the center point ($r = 0, z = L$) has been determined analytically, (Ref. 12, 13). The temperature at point p (Fig. 11C) is given by equation A-2 evaluated at $x = 0$. It will be designed ϕ_C. These three solutions will now be combined by using the inverse convolution to obtain the T_p, (Fig. 10).

The time-variable heat flux passing through the interface between the heat sink and the thermocouple is the key to the solution; this heat flux will be designated $\dot{q}_p(t)$. The temperature at point p of the thermocouple (Fig. 11C) is then given by

$$T_p(t) = \int_0^t \dot{q}_p(\lambda) \frac{\partial \phi_C(t-\lambda)}{\partial t} d\lambda . \tag{23a}$$

If the solutions of figure 11A and 11B are added together, the temperature at point p is given by

$$T_p(t) = T_{p\infty}(t) + \int_0^t \dot{q}_p(\lambda) \frac{\partial \phi_B(t-\lambda)}{\partial t} d\lambda . \tag{23b}$$

This addition may be done since the boundary conditions are compatible and the governing differential equation is linear. Now the temperature T_p given by

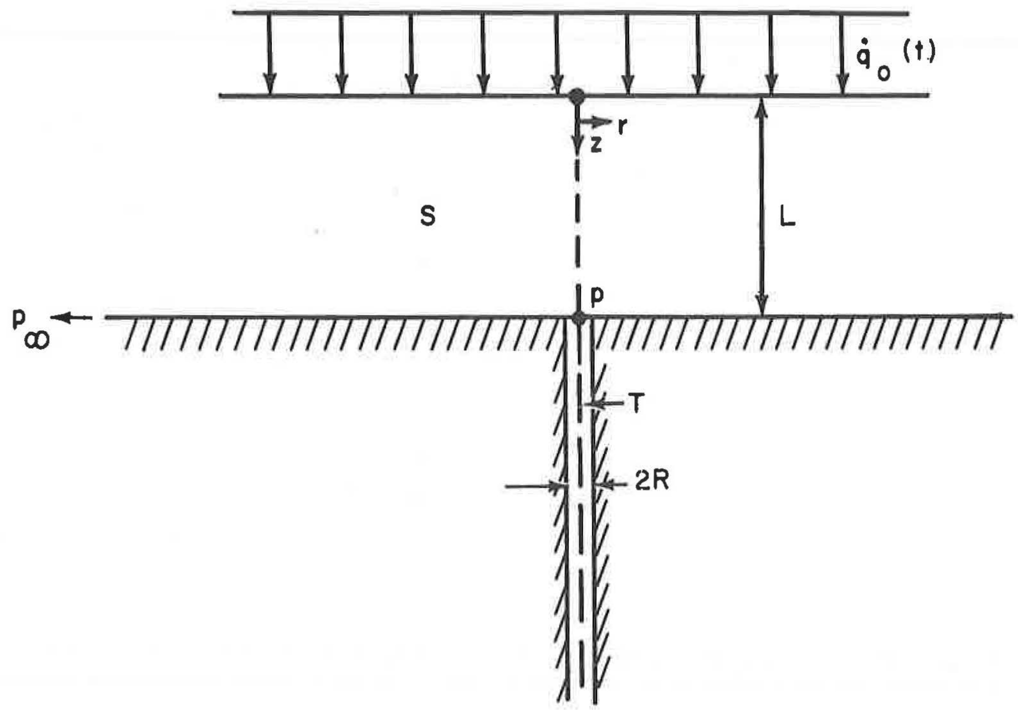

Figure 10 GEOMETRY OF COMPOSITE MATERIALS WHICH MAY BE SOLVED USING INVERSE CONVOLUTION
61-1210

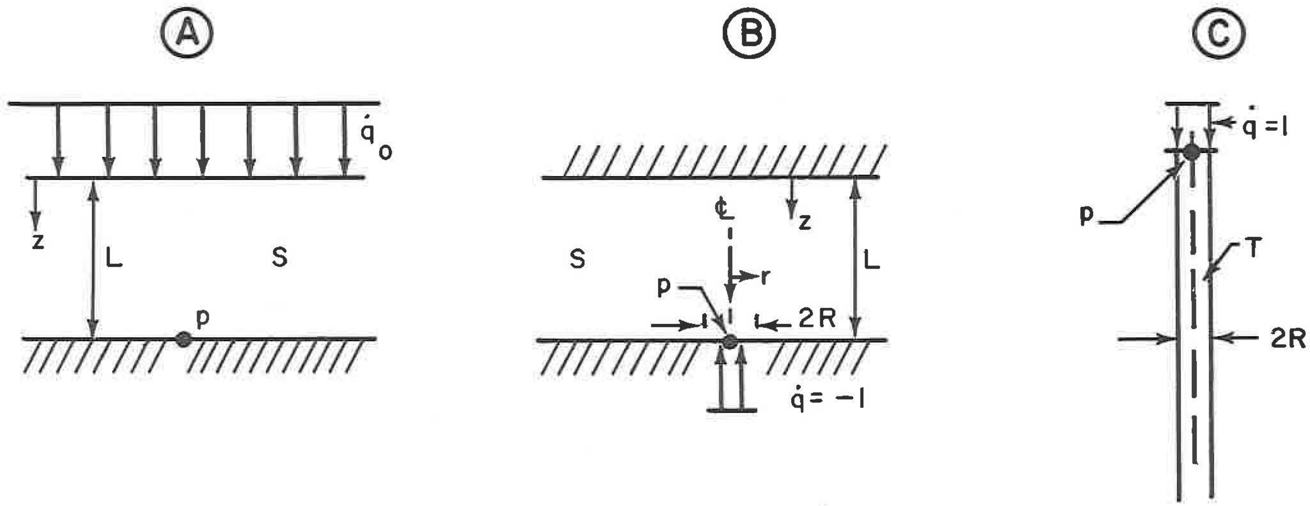

Figure 11 AUXILIARY PROBLEMS REQUIRED FOR FIGURE 9
61-1211

equation (23a) is equal to the temperature T_p from equation (23b), and the heat flux leaving the heat sink is equal to that entering the thermocouple. Hence, the boundary conditions at the sink-thermocouple interface are satisfied. It remains now to calculate $\dot{q}_p$ and T_p. One may combine equations (23a) and (23b) to obtain

$$T_{p\infty}(t) = \int_0^t \dot{q}_p(\lambda) \frac{\partial \left[\phi_B(t-\lambda) + \phi_C(t-\lambda)\right]}{\partial t} d\lambda . \tag{24}$$

The only unknown in equation (24) is $\dot{q}_p(\lambda)$ which may be calculated using the inverse convolution. $T_p(t)$ is then obtained by using $\dot{q}_p(\lambda)$ in equation (23a). This forward convolution process has also been programmed and is called 702.02. It is described in reference 1.

This problem is naturally only one of many such heat conduction problems which may be solved using the inverse convolution. This method of solution, whenever applicable, provides a technique to solve certain conduction problems which would otherwise be solved at considerably greater expense using a numerical transient two-dimensional heat conduction program.[1]

More difficult conduction problems may be solved by combining the well-known finite difference methods[5] and the inverse convolution procedure. It may be of particular value when part of the system extends to large physical distances.

The use of the inverse convolution is limited to cases for which the thermal properties may be assumed to be constant with temperature. The concepts used in deriving the inverse convolution equation may however be used to treat the temperature-variable properties case by using finite difference equations as in references 3 and 4. An investigation of a method of doing this has begun.[14] Preliminary results indicate an improvement over the methods suggested by Klamkin.[3]

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APPENDIX A

TEMPERATURE IN A SEMI-INFINITE SOLID HEATED WITH A CONSTANT HEAT FLUX

The transient temperature T in a semi-infinite solid which is heated by a constant heat flux $\dot{q}_o$ is given by[6]

$$T - T_i = \frac{2\dot{q}_o}{k} \sqrt{\alpha t} \;\text{ierfc}\; \frac{x}{2\sqrt{\alpha t}} \;. \tag{A-1}$$

The initial temperature is T_i, k is thermal conductivity, and α is thermal diffusivity. Equation A-1 may be written in a dimensionless form as

$$\theta = \frac{(T - T_i)k}{\dot{q}_o R} = 2\tau^{1/2} \;\text{ierfc}\; \frac{x/R}{2\tau^{1/2}} \;. \tag{A-2}$$

where

$$\tau = \frac{\alpha t}{R^2} \;.$$

If $x = 2R$, one obtains

$$\theta_1 = 2\tau^{1/2} \;\text{ierfc}\; \tau^{-1/2} \;, \tag{A-3}$$

and if $x = 4R$, one obtains

$$\theta_2 = 2\tau^{1/2} \;\text{ierfc}\; 2\tau^{-1/2} \;. \tag{A-4}$$

The heat flux at any point in the body is given by

$$\frac{\dot{q}}{\dot{q}_o} = \text{erfc}\; \frac{x/R}{2\tau^{1/2}} \;. \tag{A-5}$$

At $x/R = 2$, the heat flux is then

$$\frac{\dot{q}}{\dot{q}_o} = \text{erfc}\, \tau^{-1/2} \; . \qquad \text{(A-6)}$$

APPENDIX B

TEMPERATURE IN A SEMI-INFINITE SOLID WITH CONSTANT SURFACE TEMPERATURE

The transient temperature T in a semi-infinite solid which has its surface temperature suddenly increased to T_o is given by

$$T - T_i = (T_o - T_i) \operatorname{erfc} \frac{x/R}{2\tau^{1/2}} . \qquad \text{(B-1)}$$

The temperature at a point $x/R = 2$ is given by

$$\frac{T - T_i}{T_o - T_i} = \operatorname{erfc} \tau^{-1/2} . \qquad \text{(B-2)}$$

The heat flux at the surface $(x = 0)$ is given by

$$\frac{\dot{q} R}{k T_o} = \frac{\tau^{-1/2}}{\sqrt{\pi}} . \qquad \text{(B-3)}$$

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